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The vertical attenuation of irradiance as a function of the optical properties of the water

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Abstract

Average vertical attenuation coefficients, $K(av)$, for irradiance calculated by linear regression of ln $E(z)$ on z through the euphotic zone or from two irradiance values in a certain depth interval, are useful but somewhat arbitrary procedures for estimating these important apparent optical properties of the ocean. A more fundamental approach is to calculate an irradiance-weighted coefficient, ${}^wK(av)$, integrated over the whole water column, in which for each increment of depth, the corresponding irradiance value is used to weight the estimate of the irradiance coefficient in accordance with ${}^wK(av) = \int_0^\infty K(z)E(z)\,dz/\int_0^\infty E(z)\,dz$. Attenuation coefficients calculated in this way exhibit some interesting relationships both to certain other properties of the light field and to the inherent optical properties of the water. In particular, I find that for all types of irradiance ${}^wK(av) = E(0)/\int_0^\infty E(z)\,dz$, where $E(0)$ is the value of irradiance just below the water surface. For net downward irradiance, ${}^wK_E(av) = a/\bar{\mu}_c$ and ${}^wK_E(av) \approx (a/\bar{\mu}_o)[1 + (b/a)(1 - \bar{\mu}_s)]$, where $\bar{\mu}_c$ is the integral average cosine for the water column, $\bar{\mu}_o$ is the average cosine of the incoming flux just below the water surface, a is the absorption coefficient, b is the scattering coefficient, and $\bar{\mu}_s$ is the average cosine (asymmetry factor) of the scattering phase function. For scalar irradiance, ${}^wK_o(av) = a/\bar{\mu}(0)$, where $\bar{\mu}(0)$ is the average cosine of the light field just below the water surface. The extent to which these and conventionally calculated attenuation coefficients reproduce the depth variation of irradiance is explored using Monte Carlo modeling.

The value of the vertical attenuation coefficient for irradiance, averaged over some depth interval of interest—typically the euphotic zone—is the most frequently measured and useful of the set of apparent optical properties (AOP) of the ocean. There are, however, several different versions of, and correspondingly different ways of measuring, this attenuation coefficient, and it is not always clear which is the most appropriate one to use. I consider here some of these possibilities and, in particular, examine the relationship between irradiance attenuation coefficients and certain other properties of the light field, as well as the inherent optical properties (IOP) of the aquatic medium.

Theory

In the sea or lakes or any other surface waterbody, the irradiance of the solar radiation field—whether downward (E_d), upward (E_u), net downward ($\vec{E} = E_d - E_u$), or scalar (E_o)—diminishes in an approximately exponential manner with depth (z) in accordance with Eqs. 1, 2 (*see Table 1 for symbols used in the text*).

$$E(z) \simeq E(0)\exp(-K[av, 0 \rightarrow z]z) \quad \text{or} \tag{1}$$

$$\ln E(z) \simeq -K[av, 0 \rightarrow z]z + \ln E(0) \tag{2}$$

K, the vertical attenuation coefficient (m^{-1}) for the appropriate irradiance, is K_d, K_u, K_E, or K_o, for downward, upward, net, or scalar irradiance, respectively. At any specific depth, K has a precise, but localized, value given by Eq. 3.

$$K(z) = -\frac{1}{E(z)}\frac{dE(z)}{dz} \quad \text{or} \tag{3a}$$

$$K(z) = -\frac{d\ln E(z)}{dz} \tag{3b}$$

The value of $K(z)$ at any depth depends not only on the absorption and scattering properties of the water, but also on the angular distribution of the light field at that depth. Even in well-mixed water, where the inherent optical properties—absorption coefficient, a; scattering coefficient, b; scattering phase function, $\tilde{\beta}(\theta)$—are everywhere the same, the angular distribution of the light field varies with depth, typically (but not invariably) becoming more diffuse with depth as the radiant flux becomes progressively more highly scattered. This means that $K(z)$ varies with depth, but at least for narrow spectral wavebands, the variation is not great, and for any waterbody it is useful to calculate a depth-averaged value, $K(av)$, for the waveband of interest since $K(av)$, when used with Eqs. 1 or 2, summarizes in a single parameter the manner in which the irradiance is attenuated with depth in that waterbody.

There are various different ways of calculating a depth-averaged irradiance attenuation coefficient. The simplest, but least accurate, over any depth interval, z_1–z_2, is shown in Eq. 4.

$$K(av, z_1 \rightarrow z_2) = \frac{1}{(z_2 - z_1)}\ln\left[\frac{E(z_1)}{E(z_2)}\right] \tag{4}$$

A commonly used and more accurate alternative is to calculate the linear regression coefficient of ln $E(z)$ with respect to depth over the depth interval of interest.

An exact expression for depth-averaged K over any depth interval from the surface down to depth z' can be written as Eq. 5.

$$K(av, 0 \rightarrow z') = \frac{\int_0^{z'} K(z)\,dz}{z'} \tag{5}$$

Although this can, in principle, although not easily in practice, be evaluated to any desired level of accuracy (depending on the value of z used in the measurements), if I want to determine the ultimate depth-averaged K—that is, $K(av,$

Table 1. Symbols used in the text.

a	Absorption coefficient (m^{-1})
b	Scattering coefficient (m^{-1})
$\tilde{\beta}(\theta)$	Scattering phase function (sr^{-1})
E_d, E_u, $\vec{E}$, E_o	Downward, upward, net downward, and scalar irradiance (W m^{-2})
$G(\bar{\mu}_o, \bar{\mu}_s)$	Coefficient determining the relative contribution of scattering to vertical attenuation of irradiance
K_d, K_u, K_E, K_o	Vertical attenuation coefficients for downward, upward, net downward, and scalar irradiance (m^{-1})
${}^wK(av)$	The irradiance-weighted vertical attenuation coefficient for any specified type of irradiance (m^{-1})
$\bar{\mu}(z)$	Average cosine of the light field at depth z: zenith angle of all the photons in an infinitesimal volume element at depth z
$\bar{\mu}_c$	Integral average cosine of the underwater light field
$\bar{\mu}_o$	Average cosine of the incoming flux of photons immediately after passing down through the air/water surface
$\bar{\mu}(0)$	Average cosine of all the photons present just below the air/water surface
$\bar{\mu}_s$	Average cosine of scattering, or asymmetry factor, of the scattering phase function
z	Depth (m)
z_m	Midpoint of zone in which E_d falls to 1% of subsurface value—corresponds to E_d = 10% of subsurface value (m)

$0 \rightarrow \infty$)—then I immediately arrive at the position that the K value so obtained is necessarily equal to K for the asymptotic region, K_∞. Although K_∞ is an optical property of great interest in itself, its value is likely to be significantly different from that of K in the upper regions of the water column where most of the solar radiant energy is to be found and has its effects.

Irradiance-weighted average attenuation coefficients—For most oceanographic and limnological purposes, irradiance K values are more relevant if they apply to that part of the water column in which most of the attenuation of solar energy actually takes place. A common way of achieving this objective in studies of primary production is to use $K(av)$ values calculated for the euphotic zone—the zone in which nearly all the photosynthesis takes place and defined, as a rule-of-thumb, to be that layer of water from the surface down to the depth at which downward irradiance is 1% of that at $z = 0$. Typically, $K(av)$ is calculated by linear regression from ln $E(z)$ values over this range of depth.

To calculate an average K value by linear regression using irradiance values from the surface down to the 1% level is entirely appropriate for most oceanographic or limnological purposes. There is, however, another approach that also meets the criterion of yielding K values applicable to that part of the water column where most of the energy is attenuated. In this approach, the irradiance values themselves are used to weight the estimates of the irradiance attenuation coefficients. By analogy with Eq. 5, for an irradiance-weighted average vertical attenuation coefficient, I write

$$ {}^wK(av) = \frac{\int_0^\infty K(z)E(z)\,dz}{\int_0^\infty E(z)\,dz} \qquad (6) $$

where $E(z)$ can be $E_d(z)$, $E_u(z)$, $\vec{E}(z)$, or $E_o(z)$ and $K(z)$ can be $K_d(z)$, $K_u(z)$, $K_E(z)$, or $K_o(z)$, respectively. The w superscript is here used to indicate irradiance attenuation coefficients for which in the averaging by integrating, over depth, at every depth the localized value of $K(z)$ is weighted by the appropriate value of the relevant type of irradiance at that depth. The integrated product of $K(z)$ and $E(z)$ over all depths is divided by the integrated irradiance over all depths. Even though this procedure necessarily gives most weight to the upper region of the water column, where irradiance values are highest, there is nevertheless no arbitrary singling out of any particular layer since integration is carried out from the surface to infinity; that is, it is a true averaging over all depths. In practice, of course, it is only necessary to carry out integration down to depths where light levels become insignificant.

A simplification of Eq. 6 can be achieved by substituting for $K(z)$, from Eq. 3a.

$$ {}^wK(av) = -\frac{\int_0^\infty \left[\frac{1}{E(z)}\frac{dE(z)}{dz}\right]E(z)\,dz}{\int_0^\infty E(z)\,dz} = -\frac{\int_{E(0)}^{E(\infty)} dE(z)}{\int_0^\infty E(z)\,dz} $$

By integrating the numerator between 0 and infinity I get Eq. 7.

$$ {}^wK(av) = \frac{E(0)}{\int_0^\infty E(z)\,dz} \qquad (7) $$

Once again, $E(z)$ can be downward, upward, net, or scalar irradiance, and ${}^wK(av)$ can be ${}^wK_d(av)$, ${}^wK_u(av)$, ${}^wK_E(av)$, or ${}^wK_o(av)$, respectively. In principle, Eq. 7 offers a simple way of evaluating ${}^wK(av)$ from field data using a relationship as Eq. 8.

$$ {}^wK(av) = \frac{E(z_1)}{\sum_{i=1}^{N} E(z_i)\Delta z} \qquad (8) $$

Δz is a convenient increment of depth; $z_1, z_2, \ldots z_i, \ldots z_N$ are the midpoints of the first, second, . . . ith, . . . Nth depth intervals; and z_N is a depth where $E(z)$ becomes trivially small (e.g., <1.0, or 0.1% of $E[z_1]$). In practice, however, Eqs. 7 and 8, except under very calm conditions, would not be easy to use because of the notorious difficulty in measuring irradiance values just below the surface, resulting from wave-induced disturbance of the light field, a problem that bedevils field measurement of any kind of irradiance K. The solution in the present situation would be to calculate $E(z_1)$ from the more stable $E(z)$ values at greater depth, using a value of $K(z)$ estimated within the appropriate depth range.

To evaluate ${}^wK(av)$ from field data by means of Eq. 6, I can use a relationship along the lines of Eq. 9.

$$ {}^wK(av) = \frac{\sum_{i=1}^{N} K(z_i, z_{i+2})E(z_{i+1})}{\sum_{i=1}^{N} E(z_{i+1})} \tag{9} $$

To achieve adequate smoothing of the data, each $K(z)$ value is calculated from

$$ K(z_i, z_{i+2}) = \frac{1}{2\Delta z}\ln\frac{E(z_i)}{E(z_{i+2})} \tag{10} $$

(i.e., from the diminution in irradiance over two depth intervals). Each $K(z_i, z_{i+2})$ is then multiplied by $E(z_{i+1})$, the irradiance in the depth interval in between, in the numerator of Eq. 9. When making a decision about what kind of K to measure in a particular waterbody—that is, ${}^wK(av)$ or some other $K(av)$—you need to take into account not only the information contained in that attenuation coefficient, but also to what extent errors of measurement of the light field propagate through to inaccuracy in the derived values of K. Determination of ${}^wK(av)$ using Eq. 6 has the advantage that it is not so crucially dependent on the accurate measurement of one parameter, $E(0)$, as is the case with Eq. 7. Values of ${}^wK(av)$ calculated using the algorithm in Eq. 9 use data obtained through the whole illuminated water column and should yield K values of reliability comparable to those obtained by, for example, linear regression.

Irradiance-weighted average attenuation coefficient for net downward and scalar irradiance—An irradiance-weighted average attenuation coefficient can be calculated for any of the different kinds of irradiance. For net downward irradiance, however, ${}^wK(av)$ turns out to have some interesting properties. I will begin with Eq. 6 applied to net downward irradiance, $\vec{E}(z)$.

$$ {}^wK_E(av) = \frac{\int_0^\infty K_E(z)\vec{E}(z)\,dz}{\int_0^\infty \vec{E}(z)\,dz} \tag{11} $$

I assume that the water is homogeneous with inherent optical properties uniformly distributed along the vertical axis. I also ignore any contribution from Raman scattering. Using the Gershun relationship

$$ K_E(z) = \frac{a}{\bar{\mu}(z)} \tag{12} $$

where $\bar{\mu}(z)$ is the average cosine of the light field at depth z, and

$$ \bar{\mu}(z) = \frac{\vec{E}(z)}{E_o(z)} \tag{13} $$

then substituting first for $K_E(z)$, and then for $\bar{\mu}(z)$, I obtain Eq. 14.

$$ {}^wK_E(av) = \frac{a\int_0^\infty E_o(z)\,dz}{\int_0^\infty \vec{E}(z)\,dz} = a\left[\frac{\int_0^\infty \vec{E}(z)\,dz}{\int_0^\infty E_o(z)\,dz}\right]^{-1} \tag{14} $$

The expression within the square brackets was shown previously (Kirk 1999) to be equal to $\bar{\mu}_c$, the integral average cosine of the underwater light field, this being the average cosine of all the photons instantaneously present in the water column.

$$ \bar{\mu}_c = \frac{\int_0^\infty [E_d(z) - E_u(z)]\,dz}{\int_0^\infty E_o(z)\,dz} \tag{15} $$

I can therefore write

$$ {}^wK_E(av) = \frac{a}{\bar{\mu}_c} \tag{16} $$

which can be regarded as an analog, for the whole water column, of the Gershun equation (Eq. 12).

It was shown earlier (Kirk 1999) that for waterbodies in which the loss of upward-scattered photons through the surface is small enough to be disregarded, the value of $\bar{\mu}_c$ for any given incoming flux of photons with average cosine $\bar{\mu}_o$ is entirely determined by the inherent optical properties of the water in accordance with

$$ \bar{\mu}_c = \frac{\bar{\mu}_o}{1 + \frac{b}{a}(1 - \bar{\mu}_s)} \tag{17} $$

where $\bar{\mu}_s$ is the average cosine of scattering, or asymmetry factor, defined by Eq. 18.

$$ \bar{\mu}_s = \int_{4\pi} \tilde{\beta}(\theta)\cos\theta\,d\omega \tag{18} $$

Substituting for $\bar{\mu}_c$ in Eq. 16, I obtain Eq. 19.

$$ {}^wK_E(av) \approx \frac{a}{\bar{\mu}_o}\left[1 + \frac{b}{a}(1 - \bar{\mu}_s)\right] \tag{19} $$

It is of interest that Eq. 19 is rather similar in form to an empirical equation arrived at earlier (Kirk 1981, 1984, 1991) for the vertical attenuation coefficient of downward irradiance

$$ K_d(av) = \frac{a}{\bar{\mu}_o}\left[1 + \frac{b}{a}G(\bar{\mu}_o, \bar{\mu}_s)\right]^{1/2} \tag{20} $$

where the coefficient $G(\bar{\mu}_o, \bar{\mu}_s)$ is a function both of the average cosine of the refracted incident photons ($\bar{\mu}_o$) and the average cosine of scattering. This equation works equally well for $K_d(av)$, where averaging is carried out over the layer extending down to the 1% light level, and for $K_d(z_m)$, the localized value of K_d at the midpoint of this layer. Equation 20 was arrived at by curve-fitting to computer-generated light field data for a wide variety of optical water types. Like Eq. 20, Eq. 19 is an expression for an apparent optical property of the water, ${}^{w}K_E(av)$, as a function of the inherent optical properties of the water—a, b, and $\bar{\mu}_s$. Unlike Eq. 20, however, Eq. 19 was arrived at entirely on the basis of radiative transfer theory. Note that although there are theoretical differences between K_d and K_E, their actual numerical values are not far apart, as shall be seen later.

Proceeding now to the irradiance-weighted average attenuation coefficient for scalar irradiance, in accordance with Eq. 7 I can write Eq. 21.

$$ {}^{w}K_o(av) = \frac{E_o(0)}{\int_0^\infty E_o(z)\, dz} \tag{21} $$

The average cosine of the light field just below the surface is

$$ \bar{\mu}(0) = \frac{\vec{E}(0)}{E_o(0)} \tag{22} $$

giving Eq. 23.

$$ E_o(0) = \frac{\vec{E}(0)}{\bar{\mu}(0)} \tag{23} $$

Note that $\bar{\mu}(0)$, the average cosine of the light field just below the surface, is not identical to $\bar{\mu}_o$, the average cosine of the incoming flux of photons immediately below the surface. However, $\bar{\mu}(0) \rightarrow \bar{\mu}_o$ as the proportion of photons escaping through the surface tends to zero. From Eq. 15 I can write Eq. 24.

$$ \int_0^\infty E_o(z)\, dz = \frac{1}{\bar{\mu}_c} \int_0^\infty \vec{E}(z)\, dz \tag{24} $$

Substituting into Eq. 21 I obtain

$$ {}^{w}K_o(av) = \frac{\bar{\mu}_c}{\bar{\mu}(0)} \frac{\vec{E}(0)}{\int_0^\infty \vec{E}(z)\, dz} \tag{25} $$

from which, again using Eq. 7, I arrive at

$$ {}^{w}K_o(av) = \frac{\bar{\mu}_c}{\bar{\mu}(0)} {}^{w}K_E(av) \tag{26} $$

which in turn, by means of Eq. 16, leads to

$$ {}^{w}K_o(av) = \frac{a}{\bar{\mu}(0)} \tag{27} $$

another analog of the Gershun equation.

Numerical tests of the various irradiance-weighted ${}^{w}K(av)$ functions—Calculation procedure: Of the various relationships between irradiance-weighted attenuation coefficients and other parameters of the light field which I arrived at in the foregoing theoretical section, some—such as Eqs. 7, 16, 26, and 27—are exact and should be obeyed precisely. Equation 19, however, although fundamentally derived, is limited in its precision by the accuracy of the assumption, made in the original derivation of Eq. 17 (Kirk 1999), that the loss of upward-scattered photons through the surface is small enough to be disregarded. How closely this assumption corresponds to reality will vary from one waterbody to another: It should apply reasonably well throughout most of the world's oceans in which reflectance is low but poorly in turbid lakes and estuaries.

To test the validity of the above theoretical treatment, I can compare values of the various light field parameters predicted by Eqs. 7, 16, 19, and 27 with those found in simulated underwater light fields obtained by numerical modeling. For this purpose, I have used data generated by Monte Carlo modeling for waters of varying optical types. For simplicity, a flat water surface and a parallel incident light flux have been assumed. As before (Kirk 1991), the scattering phase functions used in the calculations were those measured by Petzold (1972) in a range of water types from clear oceanic to somewhat turbid harbor water, together with waters given various durations of filtration in the laboratory. The average cosine of scattering, or asymmetry factor, for each water type was obtained from

$$ \bar{\mu}_s = 2\pi \int_0^\pi \tilde{\beta}(\theta)\cos\,\theta\,\sin\,\theta\, d\theta \tag{28} $$

using the Petzold $\tilde{\beta}(\theta)$ values at 5° intervals from 0 to 180°. Table 2 lists the waters and their $\bar{\mu}_s$ values.

For all the calculations, the absorption coefficient was assigned a value of 1.0 m^{-1}, and the value of the scattering coefficient was varied as indicated. ${}^{w}K_E(av)$ as defined by Eq. 6 was calculated from the net downward irradiance values as a function of depth using Eqs. 9 and 10. ${}^{w}K_E(av)$, ${}^{w}K_o(av)$, and ${}^{w}K_d(av)$ values as specified by Eq. 7 were calculated using Eq. 8. The integral average cosine, μ_c, for each light field was calculated from the appropriate irradiance values using Eq. 15 as described in Kirk (1999), and the corresponding value of ${}^{w}K_E(av)$ as predicted by Eq. 16 was then determined. ${}^{w}K_E(av)$ as a function of the inherent optical properties of the water, arising out of the theoretical treatment of Kirk (1999) was calculated with Eq. 19. Finally, for each light field, the average cosine just beneath the surface, $\bar{\mu}(0)$, was obtained from $\vec{E}(0)/E_o(0)$ and used to provide another value of ${}^{w}K_o(av)$ as predicted by Eq. 27.

Results

I shall consider first the findings for ${}^{w}K_E(av)$, the irradiance-weighted average attenuation coefficient for net downward irradiance. The data in the Eq. 6 column in Table 2 represent, since this is the equation that defines ${}^{w}K_E(av)$, the correct values of this attenuation coefficient, to which other values can be compared. It can be seen that for all 12 water types, for scattering coefficients of both 1.0 m^{-1} and 5.0 m^{-1}, the values of ${}^{w}K_E(av)$ given by Eqs. 7 and 16 are in essentially perfect agreement with the correct (Eq. 6) values, the minor variations observed being attributable to the natural statistical variability of the raw irradiance data generated as it is by Monte Carlo simulation. These findings provide numerical confirmation of the validity of Eqs. 7 and 16.

Equation 19, expressing ${}^{w}K_E(av)$ in terms of the inherent optical properties, although not expected to be exact, works quite well for surface waters with volume scattering func-

Table 2. Vertical attenuation coefficients for irradiance in waters of different optical types. The absorption coefficient (a) has in every case been assigned a value of 1.0 m^{-1}.

Water type	$\bar{\mu}_s$	b (m^{-1})	$^wK_E(av)$ (m^{-1}) Eq. 6	$^wK_E(av)$ (m^{-1}) Eq. 7	$^wK_E(av)$ (m^{-1}) Eq. 16	$^wK_E(av)$ (m^{-1}) Eq. 19	$K_E(av)$ (m^{-1}) Lin. regr.	$^wK_d(av)$ (m^{-1}) Eq. 7	$K_d(av)$ (m^{-1}) Lin. regr.	$^wK_o(av)$ (m^{-1}) Eq. 7	$^wK_o(av)$ (m^{-1}) Eq. 27	$K_o(av)$ (m^{-1}) Lin. regr.
1. San Diego Harbor	0.922	1.0	1.071	1.071	1.072	1.078	1.108	1.070	1.107	1.028	1.027	1.086
		5.0	1.339	1.332	1.340	1.389	1.476	1.323	1.472	1.146	1.145	1.408
2. Bahama Islands Sta. 7	0.915	1.0	1.074	1.074	1.075	1.085	1.108	1.074	1.108	1.028	1.027	1.087
		5.0	1.354	1.347	1.354	1.425	1.482	1.339	1.478	1.177	1.178	1.415
3. Bahama Islands Sta. 8	0.871	1.0	1.107	1.107	1.108	1.129	1.149	1.106	1.149	1.058	1.058	1.124
		5.0	1.502	1.492	1.502	1.643	1.652	1.481	1.646	1.288	1.289	1.575
4. Bahama Islands Sta. 9	0.885	1.0	1.097	1.096	1.097	1.115	1.136	1.096	1.136	1.050	1.050	1.112
		5.0	1.455	1.446	1.455	1.574	1.597	1.436	1.592	1.253	1.253	1.521
5. Southern California, offshore Sta. 5	0.947	1.0	1.047	1.048	1.048	1.053	1.073	1.048	1.073	1.019	1.018	1.057
		5.0	1.232	1.227	1.233	1.266	1.340	1.223	1.338	1.093	1.094	1.282
6. Southern California, offshore Sta. 11	0.947	1.0	1.048	1.048	1.048	1.053	1.075	1.048	1.075	1.019	1.018	1.058
		5.0	1.232	1.227	1.233	1.267	1.340	1.223	1.338	1.093	1.094	1.282
7. Filtered freshwater	0.727	1.0	1.209	1.208	1.210	1.273	1.257	1.207	1.256	1.150	1.149	1.227
		5.0	1.925	1.910	1.925	2.367	2.095	1.891	2.085	1.675	1.675	2.002
8. Filtered freshwater + $Al(OH)_3/Mg(OH)_2$	0.926	1.0	1.067	1.067	1.068	1.074	1.105	1.067	1.105	1.025	1.030	1.082
		5.0	1.322	1.315	1.323	1.371	1.464	1.307	1.461	1.132	1.133	1.390
9. Seawater	0.934	1.0	1.061	1.061	1.061	1.066	1.095	1.060	1.094	1.022	1.021	1.073
		5.0	1.293	1.286	1.293	1.330	1.423	1.279	1.419	1.113	1.114	1.357
10. Seawater filtered 40 min	0.931	1.0	1.063	1.063	1.063	1.069	1.096	1.062	1.096	1.024	1.024	1.075
		5.0	1.302	1.295	1.302	1.345	1.431	1.288	1.428	1.126	1.126	1.363
11. Seawater filtered 100 min	0.918	1.0	1.071	1.071	1.071	1.082	1.103	1.070	1.103	1.034	1.033	1.083
		5.0	1.338	1.331	1.339	1.408	1.464	1.325	1.461	1.170	1.171	1.397
12. Seawater filtered 18 h	0.660	1.0	1.259	1.256	1.260	1.340	1.318	1.254	1.317	1.182	1.183	1.283
		5.0	2.116	2.097	2.116	2.698	2.308	2.068	2.295	1.809	1.808	2.204

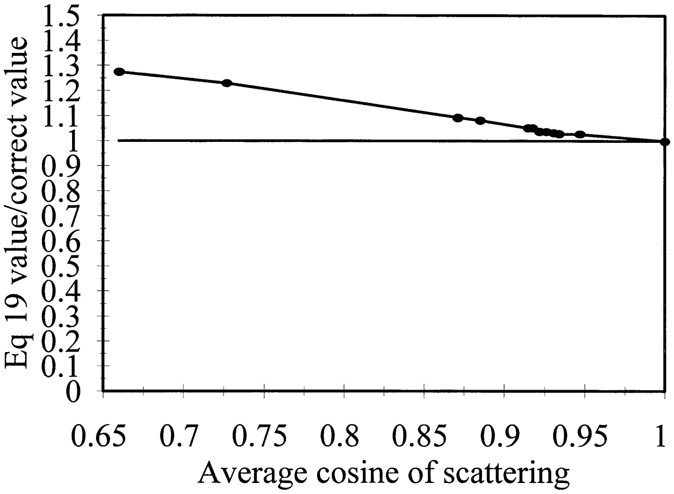

Fig. 1. Ratio of the Eq. 19 value of ${}^{w}K_E(av)$ to the correct value as a function of the average cosine of scattering for different waters. $b/a = 5.0$.

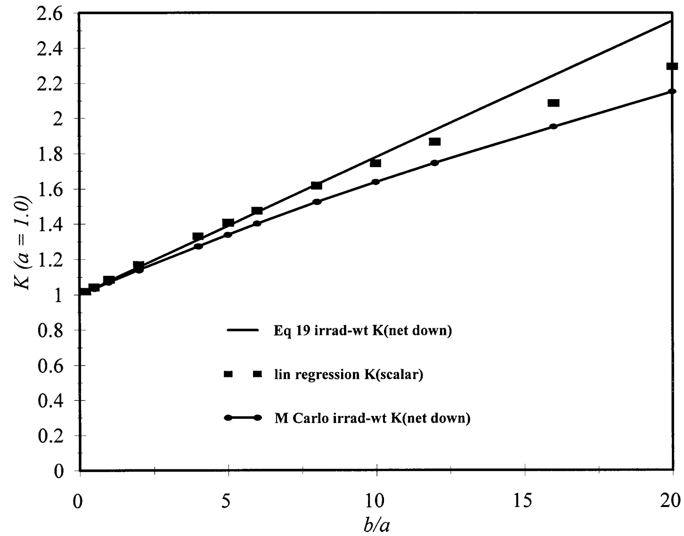

Fig. 2. Relationship between irradiance-weighted average vertical attenuation coefficient for net downward irradiance, ${}^{w}K_E(av)$, and b/a, as predicted by Eq. 19 and as observed in Monte Carlo–derived light fields. The linear regression–derived $K(av)$ for scalar irradiance obtained from the Monte Carlo data is also shown. The data are for water with the San Diego scattering phase function illuminated vertically.

tions that are likely to be encountered in the real world (water types 1–6, 9 in Table 2). Figure 1 shows, for $b/a = 5.0$, the ratio of the Eq. 19 value of ${}^{w}K_E(av)$ to the correct value as a function of the average cosine of scattering for all the waters included in this study. It can be seen that Eq. 19 tends to overestimate ${}^{w}K_E(av)$, the error increasing in an approximately linear manner as the value of $\bar{\mu}_s$ decreases. The real-world waters are clustered at the right-hand end of the graph, the error ranging from 2.8% for offshore Southern California water to 9.4% for Bahama Islands Sta. 8. For a given water type, the proportion by which the theoretical (Eq. 19) value of ${}^{w}K_E(av)$ exceeds the correct value increases with the scattering/absorption ratio (Fig. 2). For example, in water of San Diego Harbor type, the discrepancy is 0.6, 3.7, 10.8, and 18.7% for $b/a = 1.0$, 5.0, 12.0, and 20.0, respectively. This falling-off in the applicability of Eq. 19 with increasing scattering is to be expected because, in this situation, the assumption made in its derivation—that loss of upward-scattered photons through the surface is small enough to ignore—becomes progressively less realistic.

Values of the average vertical attenuation coefficient for net downward irradiance calculated by linear regression are consistently higher than the corresponding irradiance-weighted average attenuation coefficient (Eq. 6) values. For a scattering/absorption ratio of 5.0, the difference is about 10% for all the waters studied (Table 2). Similarly, linear regression–derived values of the average vertical attenuation coefficient for downward irradiance are consistently higher than the corresponding irradiance-weighted average attenuation coefficient (Eq. 7) values. For waters with $b/a = 5.0$, the difference is about 11% for all the waters studied (Table 2).

Turning to the values of ${}^{w}K_o(av)$, the irradiance-weighted vertical attenuation coefficient for scalar irradiance, note first that the values given by Eq. 27 and those given by Eq. 7 are essentially identical (Table 2), providing further numerical evidence for the validity of the theoretical treatment presented here. Once again, I find that the values calculated by linear regression are higher than the corresponding irradiance-weighted values, the difference being about twice that observed in the case of downward and net downward irradiance. For a scattering/absorption ratio of 5.0 the difference is about 21% for all the waters studied (Table 2). It seems to be a property of the irradiance-weighted coefficient for scalar irradiance that for a given value of absorption coefficient its value increases at a much slower rate with increasing scattering than do any of the other attenuation coefficients (Fig. 3). An unexpected finding is that Eq. 19, which is a theoretically derived, although approximate, expression for the irradiance-weighted attenuation coefficient for net downward irradiance as a function of inherent optical properties, does a somewhat better job at predicting the dependence of the linear regression–derived vertical attenuation coefficient for scalar irradiance on the IOP (Fig. 2). I can thus write

$$K_o(av,\ lin.\ regr.) \approx \frac{a}{\bar{\mu}_o}\left[1 + \frac{b}{a}(1 - \bar{\mu}_s)\right] \qquad (29)$$

an equation that is essentially empirical but with some indirect theoretical justification. Figure 3 shows that Eq. 19 also does quite a good job of predicting the dependence of $K_E(av,\ lin.\ regr.)$ on b/a: the same goes for $K_d(av,\ lin.\ regr.)$, which is always very close in value to $K_E(av,\ lin.\ regr.)$. Thus equations along the lines of Eq. 29 can be written for these coefficients too.

Having explored the properties of the irradiance-weighted vertical attenuation coefficients in some detail, I can now turn my attention to their usefulness as a guide to the state of the underwater light field. The main significance of any $K(av)$ function, whether irradiance-weighted or not, is that it provides information about the variation of irradiance with

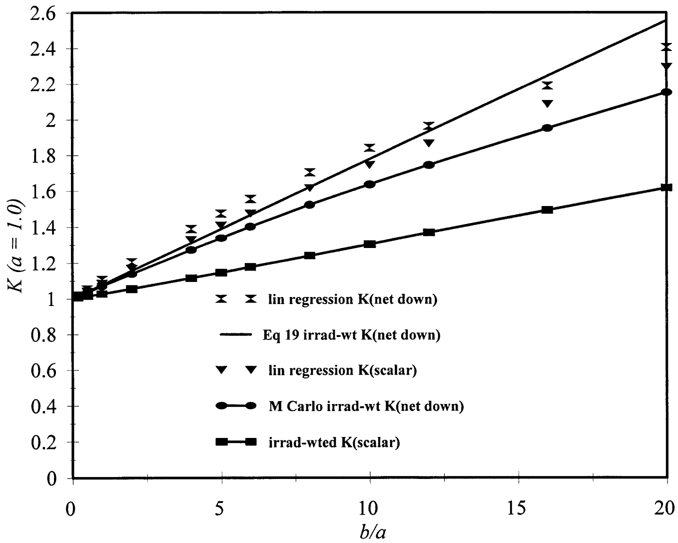

Fig. 3. $K(av)$ values for net downward and scalar irradiance as a function of the scattering/absorption ratio: $K_E(av, lin. regr.)$, ${}^wK_E(av)$ from Eq. 19, $K_o(av, lin. regr.)$, and ${}^wK_E(av)$ from Monte Carlo data. Irradiance-weighted coefficient for scalar irradiance, ${}^wK_o(av)$. The data are for water with the San Diego scattering phase function illuminated vertically.

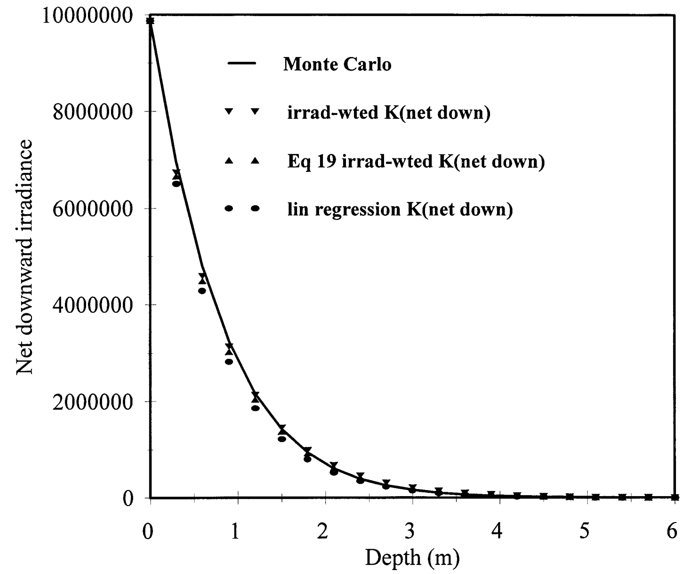

Fig. 4. Variation of net downward irradiance with depth. Water with the San Diego scattering phase function and $b/a = 4.0$ illuminated vertically. The Monte Carlo values are shown with those calculated using ${}^wK_E(av)$, ${}^wK_E(av)$ from Eq. 19), and $K_E(av, lin. regr.)$.

depth. The question to be addressed is: Which kind of $K(av)$, if I make the assumption of exponential diminution of E with z, gives the best fit to the observed curve of $E(z)$ versus z?

Downward and net downward irradiance both diminish with depth in a manner close to exponential. As a measure of how closely the exponential curve calculated with any given $K_E(av)$ reproduces the actual curve, I calculate the regression of a set of $\vec{E}(z)$ values (from just under the surface down to the depth where irradiance is ~1% of the subsurface value) obtained from the appropriate $K_E(av)$ on the corresponding Monte Carlo–derived values. With a moderate scattering/absorption ratio ($b/a = 4.0$) and vertically incident light, the linear regression–derived $K_E(av)$, the irradiance-weighted ${}^wK_E(av)$, and the theoretically derived K_E (Eq. 19) all reproduce the Monte Carlo $E(z)$ values quite well (Fig. 4), with ${}^wK_E(av)$ providing the best fit ($r^2 = 0.9992$), closely followed by K_E ($r^2 = 0.9987$, Eq. 19), and then by the linear regression–derived $K_E(av)$ ($r^2 = 0.9958$). For light incident on the surface at 45°, the three vertical attenuation coefficients have very similar values—${}^wK_E(av) = 1.478\ m^{-1}$, $K_E = 1.547\ m^{-1}$ (Eq. 20), $K_E(av, lin. regr.) = 1.541\ m^{-1}$—and all reproduce the Monte Carlo curve very closely.

In a highly scattering water ($b/a = 20.0$) illuminated vertically, it is the irradiance-weighted vertical attenuation coefficient, ${}^wK_E(av) = 2.153\ m^{-1}$, which best ($r^2 = 0.9978$) reproduces the Monte Carlo curve (Fig. 5a). The linear regression– and Eq. 19–derived coefficients (2.407 m^{-1} and 2.556 m^{-1}, respectively) both somewhat underestimate $\vec{E}(z)$ at all depths ($r^2 = 0.9920$ and 0.9853, respectively). For light incident on the surface at 45°, not only the irradiance-weighted (2.395 m^{-1}) but also the linear regression-derived (2.488 m^{-1}) coefficient accurately reproduces the Monte Carlo curve ($r^2 = 0.9996$ and 0.9990, respectively), but the Eq.

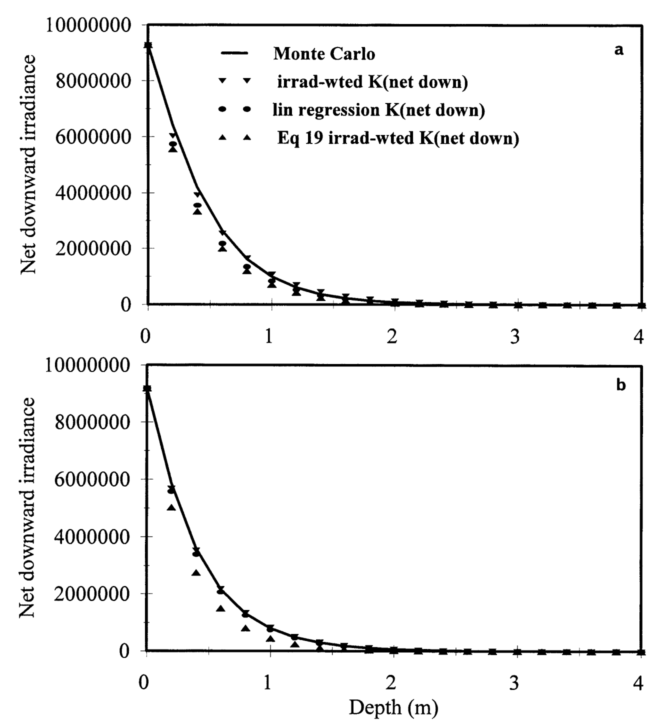

Fig. 5. Variation of net downward irradiance with depth in a highly scattering water. Water with the San Diego scattering phase function and $b/a = 20.0$ illuminated (a) vertically and (b) at 45°. The Monte Carlo values are shown with those calculated using ${}^wK_E(av)$, $K_E(av, lin. regr.)$, and ${}^wK_E(av)$ from Eq. 19.

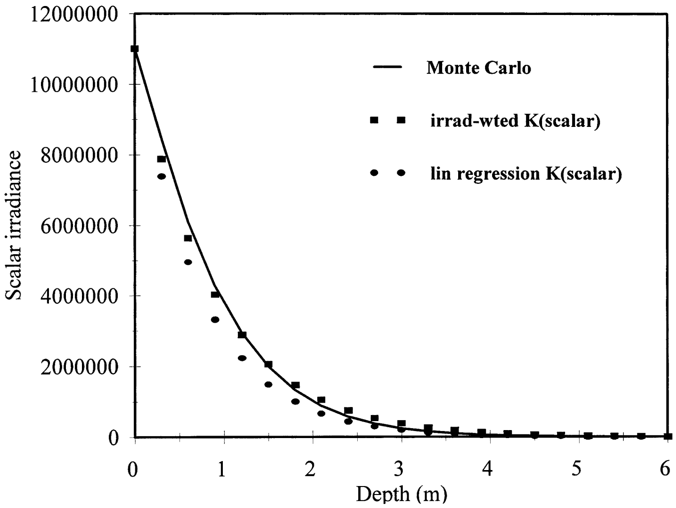

Fig. 6. Variation of scalar irradiance with depth. Water with the San Diego scattering phase function and $b/a = 4.0$ illuminated vertically. The Monte Carlo values are shown with those calculated using ${}^{w}K_o(av)$ and $K_o(av,\ lin.\ regr.)$.

19–derived coefficient (3.015 m^{-1}) once again noticeably underestimates $\vec{E}(z)$ at all depths ($r^2 = 0.9814$, Fig. 5b).

When I turn my attention to scalar irradiance, I find that the differences between the irradiance-weighted and the linear regression–derived vertical attenuation coefficients are more substantial. With a moderate scattering/absorption ratio ($b/a = 4.0$), ${}^{w}K_o(av)$ (= 1.115 m^{-1}) reproduces the Monte Carlo values more accurately than $K_o(av,\ lin.\ regr.)$ (= 1.330 m^{-1}) in the upper part of the water column where most of the radiant flux is to be found ($r^2 = 0.9960$ and 0.9839, respectively, Fig. 6). Both attenuation coefficients work somewhat better for light incident at 45°, with ${}^{w}K_o(av) = 1.339\ m^{-1}$ ($r^2 = 0.9976$) and $K_o(av,\ lin.\ regr.) = 1.518\ m^{-1}$ ($r^2 = 0.9921$).

At high scattering/absorption ratios, the variation with depth of scalar irradiance diverges quite markedly from simple exponential diminution, as shown in Fig. 7a for light vertically incident on water with $b/a = 20.0$. No single exponential attenuation coefficient can therefore satisfactorily reproduce the curve of $E_o(z)$ versus z. Nevertheless, as the figure shows, the irradiance-weighted coefficient ${}^{w}K_o(av) = 1.618\ m^{-1}$ does a somewhat better job than the linear regression–derived coefficient $K_o(av,\ lin.\ regr.) = 2.296\ m^{-1}$ ($r^2 = 0.9751$ and 0.9389, respectively). Agreement between prediction and observation improves as the incident light departs from the vertical, as can be seen in Fig. 7b for light incident at 45°. Once again the irradiance-weighted coefficient ${}^{w}K_o(av) = 1.991\ m^{-1}$ gives a somewhat better fit ($r^2 = 0.9897$) than the linear regression–derived parameter $K_o(av,\ lin.\ regr.) = 2.452\ m^{-1}$ ($r^2 = 0.9772$).

Discussion

It is clear from the data presented in this paper that irradiance-weighted vertical attenuation coefficients for irradiance provide a somewhat better guide to the variation of irradiance with depth than do coefficients calculated by the

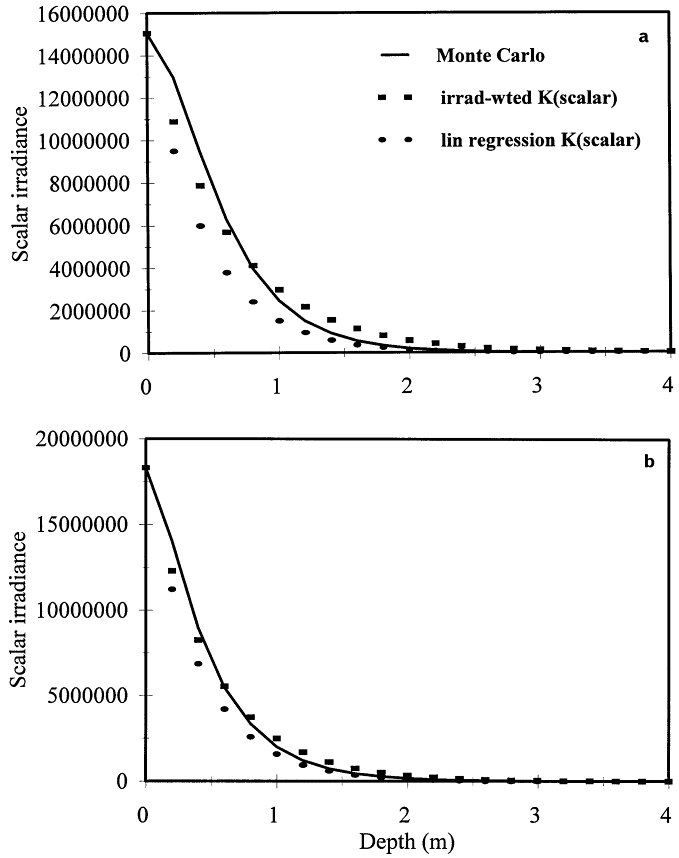

Fig. 7. Variation of scalar irradiance with depth in a highly scattering water. Water with the San Diego scattering phase function and $b/a = 20.0$ illuminated (a) vertically and (b) at 45°. The Monte Carlo values are shown with those calculated using ${}^{w}K_o(av)$ and $K_o(av,\ lin.\ regr.)$.

more conventional procedure of linear regression of ln E versus z. The differences are not, however, very great, and what is more significant and interesting about the irradiance-weighted Ks is that, when taken together with certain other properties of the field, specifically the average cosines $\bar{\mu}_c$ and $\bar{\mu}(0)$, they turn out to have fundamental relationships to an inherent optical property of the medium, namely a, in a manner analogous to that prescribed by the well-known Gershun equation between $K_E(z)$, $\bar{\mu}(z)$ and a. In addition, one of these coefficients, ${}^{w}K_E(av)$, can be shown to be related to three of the IOPs of the medium—the absorption coefficient, the scattering coefficient, and the asymmetry factor of the scattering phase function—through Eq. 19, an equation that, although not exact, arises out of fundamental radiative transfer considerations and that applies with reasonable accuracy to all except highly scattering waters. Because the linear regression–derived vertical attenuation coefficients for scalar, net downward, and downward irradiances tend to co-vary with, and are not numerically very different from, ${}^{w}K_E(av)$, Eq. 19 provides a useful approximate guide to how the values of these coefficients also can be expected to vary with a, b, and $\bar{\mu}_s$ in waters of different optical types. This exploration of the properties of irradiance-weighted vertical attenuation coefficients has thus extended our understanding of

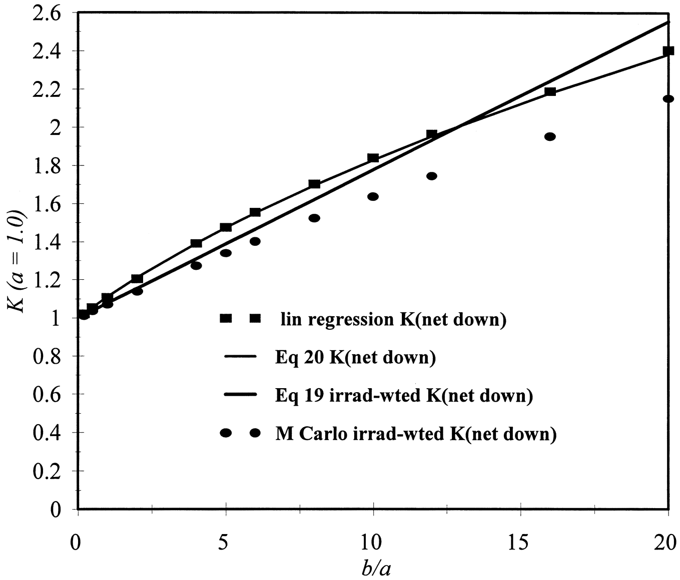

Fig. 8. Comparison of theoretical and empirical equations for vertical attenuation coefficients for net downward irradiance over a range of b/a values. The irradiance-weighted coefficient values, ${}^{w}K_E(av)$, are calculated from Monte Carlo data and from Eq. 19. $K_E(av)$ values are calculated by linear regression from Monte Carlo data and from Eq. 20. The data are for water with the San Diego scattering phase function illuminated vertically.

the relationship between the AOP and the IOP of the ocean. However, the fact remains that the entirely empirical Eq. 20, applied, for example (using $G = 0.2344$, extracted from the data, *see* Kirk 1984), to the linear regression–based K_E values as a function of b/a, works almost perfectly over the entire range of scattering, whereas Eq. 19, despite its sound theoretical basis, when applied to the irradiance-weighted ${}^{w}K_E(av)$ values, only works well over the lower third or so of the range (Fig. 8). As seen earlier (Eq. 16), ${}^{w}K_E(av)$ is inversely proportional to $\bar{\mu}_c$. The progressively increasing degree to which Eq. 19 overestimates ${}^{w}K_E(av)$ at higher scattering/absorption ratios is essentially attributable to the corresponding progressive increase in the extent to which the theoretically derived (Eq. 17) value of $\bar{\mu}_c$ underestimates the true value as b/a increases (Kirk 1999). The unreasonable effectiveness of Eq. 20 remains a puzzle.

I have noted that the performance of single attenuation coefficient values, as a tool for accurately predicting the depth profile of the light field, improves when the angle of the incident light flux departs from the vertical. The reason why a single K value cannot satisfactorily reproduce the E versus z curve is that, in waters with realistic b/a values, the angular distribution of the light field changes with depth, moving progressively toward its final, more diffuse, asymptotic shape. The alteration with depth is at its maximum when the light starts off as a vertical flux. When the light begins its downward journey through the water column at an angle already well removed from the vertical, then the change in angular distribution with depth is proportionately less, so the E versus z curve can be represented more satisfactorily by a single exponential K value.

In the theoretical treatment presented above, showing the relationship between irradiance-weighted attenuation coefficients and the IOP, the water was assumed to be optically homogeneous. In the case of ocean waters with a mixed layer underlain by a deep chlorophyll maximum near the bottom of the euphotic zone, these relationships will therefore not apply exactly. However, because irradiance weighting is carried out in the calculation of the ${}^{w}K(av)$ values, the optical character of the well-illuminated mixed layer is ensured to predominate, so in real-world situations, it will commonly be the case that these various relationships can, to a reasonable approximation, be assumed to be applicable to the mixed layer. However, where there is, for whatever reason (e.g., layering of different water types in estuaries), substantial vertical inhomogeneity in the well-illuminated zone, the equations relating the AOP to the IOP cannot validly be applied.

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